Meriam 2/6 : Polar (r-θ) coordinates (contd.)

$$\vec{r} = r\,\hat{e}_r$$

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$$\vec{a} = \text{?}$$

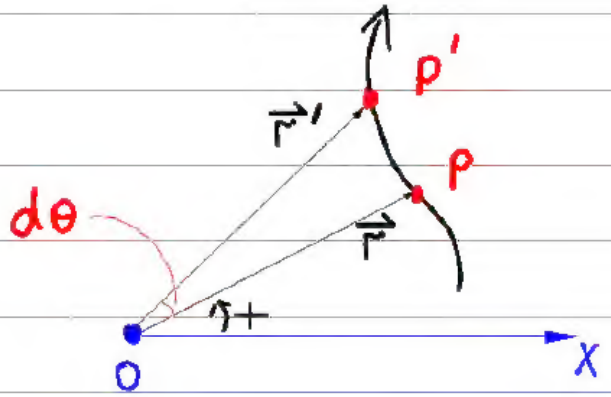
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta\right)$$

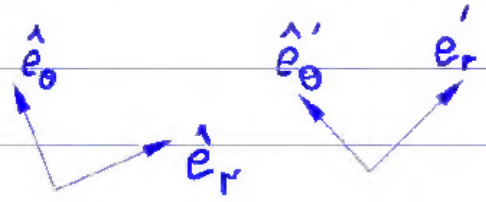
$$\vec{a} = \ddot{r}\,\hat{e}_r + \dot{r}\,\frac{d\hat{e}_r}{dt} + \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}\,\frac{d\hat{e}_\theta}{dt} \quad \star$$

$\frac{d\hat{e}_r}{dt} = \dot{\theta}\,\hat{e}_\theta$

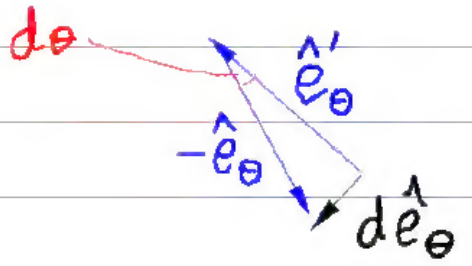
$\frac{d\hat{e}_\theta}{dt}$ = ?

$$\frac{d\hat{e}_\theta}{dt} = \lim_{\Delta t \to 0} \frac{\hat{e}_\theta' - \hat{e}_\theta}{dt}$$





$$d\theta = \frac{|d\hat{e}_\theta|}{1}$$



→ $$\frac{d\hat{e}_\theta}{dt} = (d\theta)\left(-\hat{e}_r\right) = -\dot{\theta}\,\hat{e}_r$$ substitute into ★

→

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta$$

Note: all terms have units of m/s^2

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

What does each parameter mean?

$$\vec{a} = \dot{\vec{v}} \qquad \vec{v} = \dot{\vec{r}}, \quad v = |\vec{v}|, \quad r = |\vec{r}|$$

$$r,\ \dot{r},\ \ddot{r},\ \vec{r},\ \dot{\vec{r}},\ \hat{e}_r$$

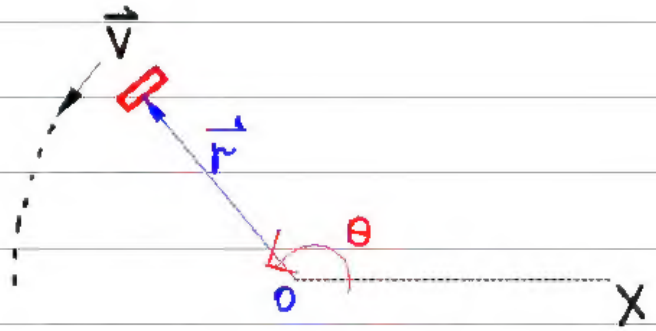
$$\theta,\ \dot{\theta},\ \ddot{\theta},\ \hat{e}_\theta$$

Note:

$v \neq \dot{r}$

$a_r \neq \ddot{r}$

$a_r \neq \dot{v}_r$



What are characteristics of spaceship's acceleration?

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta$$

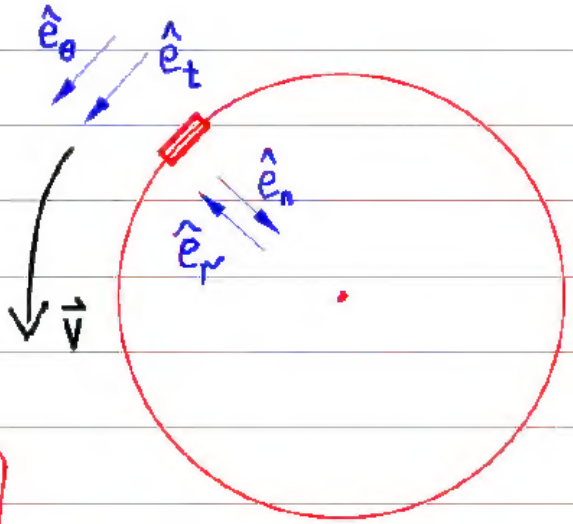
$$v_\theta = r\dot{\theta} \quad ; \quad v_r = \dot{r}$$

Which coordinate system should you use for circular motion?

~~x-y~~ ?

n-t ?

r-θ ?



r-θ: $\dot{r} = \ddot{r} = 0$

$$\vec{a} = -r\dot{\theta}^2 \hat{e}_r + r\ddot{\theta}\hat{e}_\theta$$

n-t: $$\vec{a} = \frac{v^2}{r}\hat{e}_n + \dot{v}\hat{e}_t$$

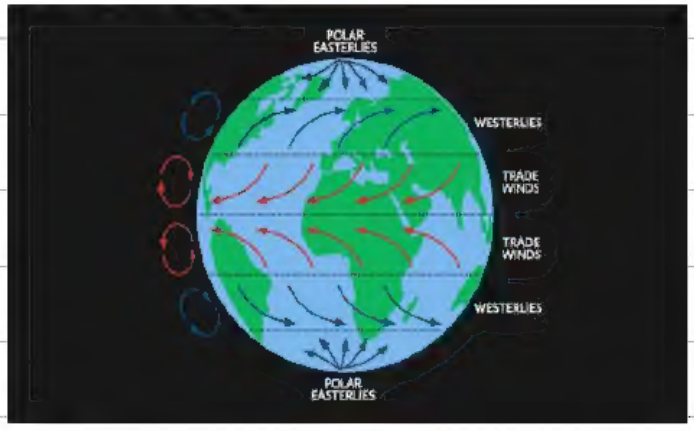
} compare for circular motion about o

For circular motion, $\hat{e}_r = -\hat{e}_n$; $v = r\dot{\theta}$; $\dot{v} = \frac{d}{dt}(r\dot{\theta}) = r\ddot{\theta}$

$$|\vec{a}| = \sqrt{a_r^2 + a_\theta^2} = \sqrt{a_n^2 + a_t^2} = \sqrt{a_x^2 + a_y^2}$$

The term $2\dot{r}\dot{\theta}$ is called the coriolis component of accelenation.

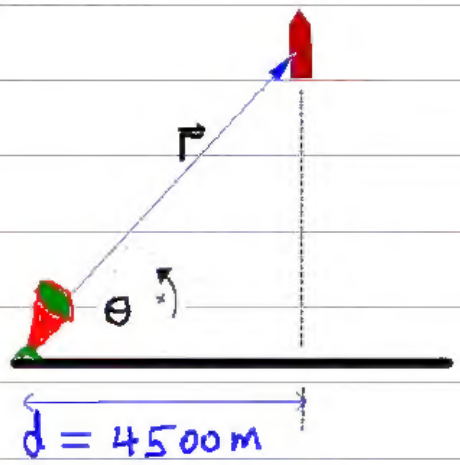
Can you give some examples?



Eg.

A rocket is fired vertically and tracked by a radar station. At the instant that $\theta = 60°$, $r = 9\ km$; $\ddot{r} = 21\ m/s^2$; $\dot{\theta} = 0.02\ s^{-1}$

Determine $\vec{a}$ when $\theta = 60°$



$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta$$

$$\rightarrow a_r = 21 - (9000)(0.02)^2$$

$$a_r = 17.4\ m/s^2$$

$\dot{r}$? consider: $\frac{d}{dt}\{\, d = r\cos\theta \,\}$

$$0 = \dot{r}\cos\theta + r(-\sin\theta)\,\dot{\theta}$$

$$\rightarrow \dot{r} = r\dot{\theta}\tan\theta$$

at $\theta = 60°$, $\dot{r} = (9000)(0.02)\tan(60°)$

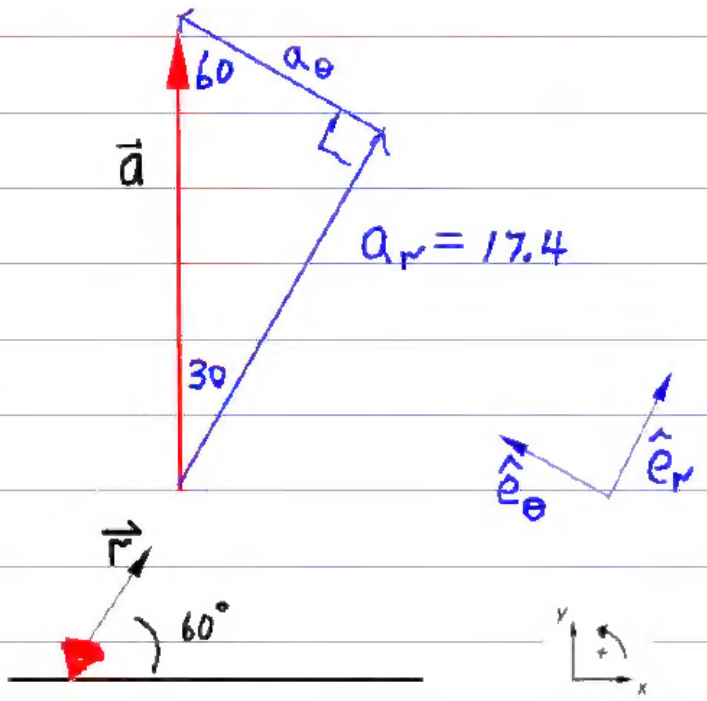
$$\dot{r} = 312\ m/s$$

$\ddot{\theta}$? We could take a derivative of $\rightarrow \dot{\theta} \rightarrow a_\theta \rightarrow \vec{a}$

But is there a better, faster, easier way?

We know that rocket motion is vertical

→ $\vec{a}$ is vertical



$|\vec{a}| \cos 30^\circ = a_r$ (0.8660 · 17.4)

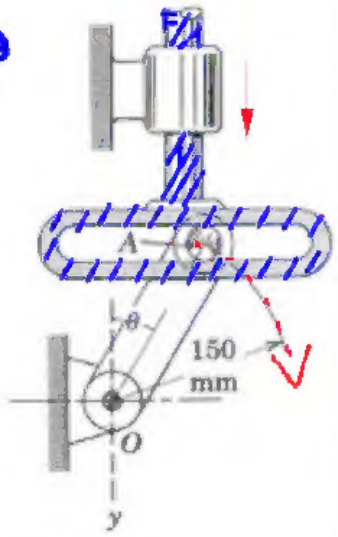
→ $\vec{a} = 20.1 \text{ m/s}^2 \, \hat{j}$

Also, $\frac{a_\theta}{a_r} = \tan 30^\circ$ (a_r = 17.4)

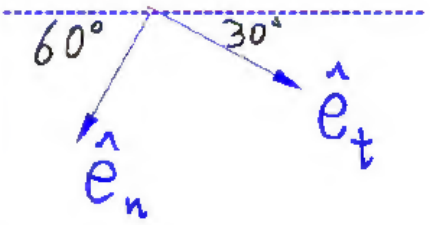
→ $a_\theta = 10.05 \text{ m/s}^2$

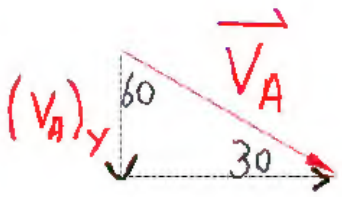
→ $\vec{a} = 10.05 \, \hat{e}_\theta + 17.4 \, \hat{e}_r$ m/s²

Rotation of crank OA is controlled by the horizontal slotted link which has a downward acceleration and velocity of 3.5 m/s² and 0.3 m/s, respectively, when $\theta = 30°$. For this instant determine the velocity and acceleration of A. Also find the corresponding values of $\dot{\theta}$ and $\ddot{\theta}$.

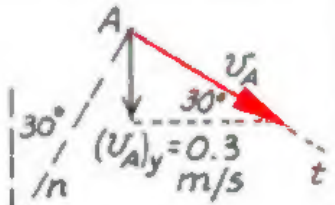


$\vec{V}_A \perp \overline{OA}$





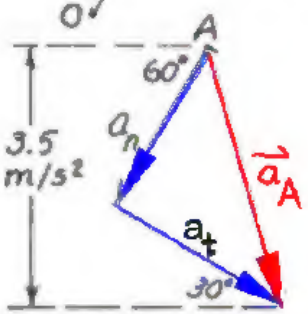
Point A has circular motion so has tangential velocity and n- and t-components of acceleration.



$v_A = (v_A)_y / \sin 30° = \frac{0.3}{0.5} = \boxed{0.6 \text{ m/s}}$

$v_A = r\dot{\theta}$, $\dot{\theta} = 0.6/0.150 = \boxed{4 \text{ rad/s}}$

$a_n = v_A^2/r = 0.6^2/0.150 = 2.4 \text{ m/s}^2$



Thus $2.4 \sin 60° + a_t \sin 30° = 3.5$

$a_t = 2.84 \text{ m/s}^2$, $a_A = \sqrt{2.4^2 + 2.84^2} = \boxed{3.72 \text{ m/s}^2}$

$a_t = r\ddot{\theta}$, $\ddot{\theta} = 2.84/0.150 = \boxed{18.95 \text{ rad/s}^2}$

JLM

Recall: For <u>circular</u> motion:

$a_t \sim a_\theta$

$a_n \sim -a_r$

r-θ: $\dot{r} = \ddot{r} = 0$

$$\vec{a} = -r\dot{\theta}^2 \hat{e}_r + r\ddot{\theta}\,\hat{e}_\theta$$

n-t:

$$\vec{a} = v\dot{\theta}\,\hat{e}_n + \dot{v}\,\hat{e}_t$$

Summary

3 coordinate systems

- $x-y$
- $n-t$
- $r-\theta$

$r-\theta$:

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta$$

For next lecture

Read Meriam 2/8: Relative Motion